

ON \mathfrak{F} -HYPERCENTRAL AND \mathfrak{F} -HYPERECCENTRIC MODULES FOR FINITE SOLUBLE GROUPS

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ABSTRACT. I prove the group theory analogues of some Lie and Leibniz algebra results on \mathfrak{F} -hypercentral and \mathfrak{F} -hypercetric modules.

1. INTRODUCTION

The theory of saturated formations and projectors for finite soluble groups was started by Gaschütz in [8], further developed by Gaschütz and Lubeseder in [9] and extended by Schunck in [10]. This theory is set out in Doerk and Hawkes [7]. The analogous theory for Lie algebras was developed by Barnes and Gastineau-Hills in [5] and for Leibniz algebras by Barnes in [4].

If \mathfrak{F} is a saturated formation of soluble Lie algebras and V, W are \mathfrak{F} -hypercentral modules for the soluble Lie algebra L , then the modules $V \otimes W$ and $\text{Hom}(V, W)$ are \mathfrak{F} -hypercentral by Barnes [1, Theorem 2.1], while if V is \mathfrak{F} -hypercentral and W is \mathfrak{F} -hypercetric, then $V \otimes W$ and $\text{Hom}(V, W)$ are \mathfrak{F} -hypercetric by Barnes [2, Theorem 2.3]. If $L \in \mathfrak{F}$ and V is an L -module, then V is the direct sum of a \mathfrak{F} -hypercentral submodule V^+ and a \mathfrak{F} -hypercetric submodule V^- by Barnes [1, Theorem 4.4]. The group theory analogues of these theorems are easily proved if we restrict attention to modules over the field \mathbb{F}_p of p elements, (the case which arises from considering chief factors of soluble groups), but the concepts are meaningful for modules over arbitrary fields of characteristic p , so I prove them in this generality.

All groups considered in this paper are finite. If V is an FG -module, I denote the centraliser of V in G by $C_G(V)$. In the following, \mathfrak{F} is a saturated formation of finite soluble groups. By Lubeseder's Theorem, (see Doerk and Hawkes [7, Theorem IV 4.6, p. 368]) \mathfrak{F} is locally defined, that is, we have for each prime p , a (possibly empty) formation $f(p)$ and \mathfrak{F} is the class of all groups G such that, if A/B is a chief factor of G of p -power order, $G/C_G(A/B) \in f(p)$. In this case, we write $\mathfrak{F} = \text{Loc}(f)$ and call it the formation locally defined by f . The formation function f is called *integrated* if, for all p , $f(p) \subseteq \text{Loc}(f)$. A saturated formation always has an integrated local definition. In this paper, I will always assume that the formation function we are using is integrated.

2. \mathfrak{F} -HYPERCENTRAL AND \mathfrak{F} -HYPERECCENTRIC MODULES

Let G be a soluble group whose order $|G|$ is divisible by the prime p , and let F be a field of characteristic p . I denote by $B_1(FG)$ the principal block of irreducible FG -modules. An irreducible FG -module V is called \mathfrak{F} -central (or (G, \mathfrak{F}) -central if I need to specify the group) if $G/C_G(V) \in f(p)$ and \mathfrak{F} -eccentric otherwise. For the special

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case of $F = \mathbb{F}_p$, V \mathfrak{F} -central is equivalent to the split extension of V by $G/\mathcal{C}_G(V)$ being in \mathfrak{F} (provided that f is integrated). A module V is called \mathfrak{F} -hypercentral if every composition factor of V is \mathfrak{F} -central. It is called \mathfrak{F} -hypereccentric if every composition factor is \mathfrak{F} -eccentric.

Theorem 2.1. *Let V, W be \mathfrak{F} -central FG -modules. Then $V \otimes W$ and $\text{Hom}(V, W)$ are \mathfrak{F} -hypercentral.*

Proof. Let $A = \mathcal{C}_G(V)$ and $B = \mathcal{C}_G(W)$. Put $C = A \cap B$. Then $V \otimes W$ and $\text{Hom}(V, W)$ are G/C -modules and $G/C \in f(p)$. It follows easily that $V \otimes W$ and $\text{Hom}(V, W)$ are \mathfrak{F} -hypercentral. \square

Proving the analogue of Barnes [2, Theorem 2.3] requires more work. I first prove the analogue of Barnes [1, Theorem 4.4] which follows easily from the following lemma.

Lemma 2.2. *Let $G \in \mathfrak{F}$. Let A, B be irreducible FG -modules and let V be an extension of A by B . Suppose one of A, B is \mathfrak{F} -central and the other \mathfrak{F} -eccentric. Then V splits over A .*

Proof. A module A is \mathfrak{F} -central if and only if its dual $\text{Hom}(A, F)$ is \mathfrak{F} -central. Hence we need only consider the case in which A is \mathfrak{F} -eccentric and $B = V/A$ is \mathfrak{F} -central.

First consider the case $F = \mathbb{F}_p$. Consider the split extension X of V by G . Since V/A is an \mathfrak{F} -central chief factor of X and $X/V \in \mathfrak{F}$, we have $X/A \in \mathfrak{F}$. But A is \mathfrak{F} -eccentric, so $X \notin \mathfrak{F}$. Therefore X splits over A and it follows that V splits over A .

Now let $\{\theta_i \mid i \in I\}$ be a basis (possibly infinite) of F over \mathbb{F}_p . We now consider V as an $\mathbb{F}_p G$ -module. Take $a \in A$, $a \neq 0$. Then $(\mathbb{F}_p G)a$ is a finite-dimensional submodule of A , so there exists a finite-dimensional irreducible $\mathbb{F}_p G$ -submodule C of A . Then $\theta_i C$ is a submodule isomorphic to C . It follows that $A = \bigoplus_{i \in I} \theta_i C = F \otimes_{\mathbb{F}_p} C$ and $G/\mathcal{C}_G(A) = G/\mathcal{C}_G(C)$. Thus C is \mathfrak{F} -eccentric. Similarly, there exists a finite-dimensional irreducible $\mathbb{F}_p G$ -submodule D of B and $B = \bigoplus_{i \in I} \theta_i D = F \otimes_{\mathbb{F}_p} D$. Also, D is \mathfrak{F} -central. It follows that $\text{Ext}^1(D, C) = 0$. But

$$\text{Ext}_{FG}^1(B, A) = F \otimes_{\mathbb{F}_p} \text{Ext}_{\mathbb{F}_p G}^1(D, C) = 0.$$

Therefore V splits over A . \square

Theorem 2.3. *Let $G \in \mathfrak{F}$ and let V be an FG -module. Then there exists a direct decomposition $V = V^+ \oplus V^-$ where V^+ is \mathfrak{F} -hypercentral and V^- is \mathfrak{F} -hypereccentric.*

Proof. Let $V_0 = 0 \subset V_1 \subset \cdots \subset V_n = V$ be a composition series of V . If for some i , we have V_i/V_{i-1} \mathfrak{F} -eccentric and V_{i+1}/V_i \mathfrak{F} -central, by Lemma 2.2, we can replace V_i by a submodule V'_i between V_{i+1} and V_{i-1} , so bringing the \mathfrak{F} -central factor below the \mathfrak{F} -eccentric factor. By repeating this, we obtain a composition series in which all \mathfrak{F} -central factors are below all \mathfrak{F} -eccentric factors. This gives us an \mathfrak{F} -hypercentral submodule V^+ with V/V^+ \mathfrak{F} -hypereccentric. Likewise, we can bring all \mathfrak{F} -eccentric factors below the \mathfrak{F} -central factors, so obtaining an \mathfrak{F} -hypereccentric submodule V^- with V/V^- \mathfrak{F} -hypercentral. Clearly, $V = V^+ \oplus V^-$. \square

For Lie and Leibniz algebras, there is a strengthened form ([3, Lemma 1.1] and [4, Theorem 3.19]) of this theorem. If U is a subnormal subalgebra of the not

necessarily soluble algebra L and V is an L -module, then the U -module components V^+, V^- are L -submodules.

Theorem 2.4. *Let $U \in \mathfrak{F}$ be a normal subgroup of the not necessarily soluble group G and let V be an FG -module. Then the (U, \mathfrak{F}) -components V^+, V^- are FG -submodules.*

Proof. Let W be either of V^+, V^- and let $g \in G$. Consider the action of $u \in U$ on gW . We have $ugW = g(g^{-1}ug)W \subseteq gW$. Thus gW is a U -submodule of V . If A is a composition factor of W , then gA is a composition factor of gW , and $\mathcal{C}_U(gA) = g\mathcal{C}_U(A)g^{-1}$. Thus $U/\mathcal{C}_U(gA) \simeq U/\mathcal{C}_U(A)$ and gA is (U, \mathfrak{F}) -central if and only if A is (U, \mathfrak{F}) -central. Thus gV^+ is (U, \mathfrak{F}) -hypercentral and so $gV^+ \subseteq V^+$ for all $g \in G$. Similarly, $gV^- \subseteq V^-$ for all $g \in G$. \square

Example 2.5. Let G be the group of permutations of the set of symbols $\{e_1, \dots, e_6\}$ generated by the permutations of $\{e_1, e_2, e_3\}$ and the permutation $(14)(25)(36)$ and let U be the subgroup generated by the permutation (123) . Then U is subnormal in G , being normal in the subgroup N consisting of those permutations which map $\{e_1, e_2, e_3\}$ into itself. Let F be a field of characteristic 2 and let V be the vector space over F with basis $\{e_1, \dots, e_6\}$. Let \mathfrak{F} be the saturated formation of all nilpotent groups. Then \mathfrak{F} is locally defined by the function $f(p) = \{1\}$ for all primes p . Considering V as U -module, we have $V^+ = \langle e_1 + e_2 + e_3, e_4, e_5, e_6 \rangle$ and $V^- = \langle e_1 + e_2, e_2 + e_3 \rangle$. (If F contains a root of $x^2 + x + 1$, then V^- is the direct sum of two non-trivial 1-dimensional modules. Otherwise, V^- is irreducible.) V^+ and V^- are invariant under the action of N but not invariant under the action of G . Thus Theorem 2.4 cannot be extended to U subnormal.

Suppose $G \in \mathfrak{F}$. Clearly, the trivial FG -module F is \mathfrak{F} -central. From Theorem 2.3, it follows that if V is an irreducible FG -module in the principal block, then V is \mathfrak{F} -central. Since $H^n(G, V) = 0$ for all n if V is not in the principal block, it follows that for any \mathfrak{F} -hypercetric module V , we have $H^n(G, V) = 0$ for all n . We cannot conclude from $H^n(G, V) = 0$ for all n that V is \mathfrak{F} -hypercetric as, for any V , there is some \mathfrak{F} for which V is \mathfrak{F} -hypercentral. To obtain a sufficient condition for V to be \mathfrak{F} -hypercetric, we use the \mathfrak{F} -cone over G .

Definition 2.6. Suppose $G \in \mathfrak{F}$. The \mathfrak{F} -cone over G is the class (\mathfrak{F}/G) of all pairs (X, ϵ) where $X \in \mathfrak{F}$ and $\epsilon : X \rightarrow G$ is an epimorphism. We usually omit ϵ from the notation, writing simply $X \in (\mathfrak{F}/G)$.

Any FG -module V is an FX -module via ϵ for any $X \in (\mathfrak{F}/G)$. Then V is \mathfrak{F} -hypercentral or \mathfrak{F} -hypercetric as FX -module if and only if it is \mathfrak{F} -hypercentral, respectively \mathfrak{F} -hypercetric as FG -module.

Theorem 2.7. *Suppose $G \in \mathfrak{F}$ and that $H^1(X, V) = 0$ for all $X \in (\mathfrak{F}/G)$. Then V is \mathfrak{F} -hypercetric.*

Proof. By Theorem 2.3, V is a direct sum of a \mathfrak{F} -hypercentral module and a \mathfrak{F} -hypercetric module. Thus, without loss of generality, we may suppose that V is \mathfrak{F} -hypercentral and we then have to prove that $V = 0$. So suppose that $V \neq 0$. There exists a minimal $\mathbb{F}_p G$ -module W of V . (W is finite-dimensional, whatever the field F .) We form the direct sum A of sufficiently many copies of W to ensure that $\dim_F \text{Hom}_{\mathbb{F}_p G}(A, V) > \dim H^2(G, V)$. Let X be the split extension of A by

G . As W is \mathfrak{F} -central, $X \in (\mathfrak{F}/G)$ and by assumption, $H^1(X, V) = 0$. We use the Hochschild-Serre spectral sequence to calculate $H^1(X, V)$. We have

$$E_2^{2,0} = H^2(X/A, V^A) = H^2(G, V)$$

and

$$E_2^{0,1} = H^0(X/A, H^1(A, V)) = \text{Hom}_{\mathbb{Z}}(A, V)^G = \text{Hom}_{\mathbb{F}_p G}(A, V).$$

Now $d_2^{0,1}$ maps $E_2^{0,1}$ into $E_2^{2,0} = H^2(G, V)$. As $\dim H^2(G, V) < \dim E_2^{0,1}$, we have $\ker(d_2^{0,1}) \neq 0$. So $E_3^{0,1} \neq 0$ and $H^1(X, V) \neq 0$ contrary to assumption. \square

Theorem 2.8. *Suppose $G \in \mathfrak{F}$. Suppose that V is an \mathfrak{F} -hypercentral FG -module and that W is an \mathfrak{F} -hypercetric FG -module. Then $V \otimes W$ and $\text{Hom}(V, W)$ are \mathfrak{F} -hypercetric.*

Proof. Let $X \in (\mathfrak{F}/G)$. Then V and W are \mathfrak{F} -hypercentral and \mathfrak{F} -hypercetric respectively as FX -modules. Every FX -module extension of W by V splits. Thus $H^1(X, \text{Hom}(V, W)) = 0$. By Theorem 2.7, $\text{Hom}(V, W)$ is \mathfrak{F} -hypercetric. The dual module $V^* = \text{Hom}(V, F)$ is also \mathfrak{F} -hypercentral. As

$$V \otimes W \simeq V^{**} \otimes W \simeq \text{Hom}(V^*, W),$$

we have also that $V \otimes W$ is \mathfrak{F} -hypercetric. \square

3. BLOCKS

Let F be a field of characteristic p and let \mathfrak{F} be a saturated formation of finite soluble groups locally defined by the function f with $f(p) \neq \emptyset$. Let $U \in \mathfrak{F}$ be a normal subgroup of the not necessarily soluble group G . Then the direct decomposition $V = V^+ \oplus V^-$ with respect to U given by Theorem 2.4 is natural. But if we take a partition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ of the set \mathcal{B} of blocks of FG -modules, we have a natural direct decomposition $V = V^+ \oplus V^-$ of FG -modules where every irreducible composition factor of V^+ is in a block in \mathcal{B}^+ and every composition factor of V^- is in a block in \mathcal{B}^- . Further, every natural direct decomposition of FG -modules has this form. Thus the (U, \mathfrak{F}) direct decomposition $V = V^+ \oplus V^-$ is the $(\mathcal{B}^+, \mathcal{B}^-)$ decomposition for some partition of \mathcal{B} . It follows that if some irreducible FU -module in an FG -block B is \mathfrak{F} -central, then all irreducibles in B are \mathfrak{F} -central. The special case of this where $U = G$ and $F = \mathbb{F}_p$ has been proved without assuming \mathfrak{F} locally defined as a stage in a proof that all saturated formations are locally definable. (See Doerk and Hawkes [7, Lemma IV 4.4].) I investigate the relationship between \mathfrak{F} and the partition $(\mathcal{B}^+, \mathcal{B}^-)$.

Lemma 3.1. *Let A/B be a p -chief factor of U and let $V = F \otimes_{\mathbb{F}_p} (A/B)$. Then V is (U, \mathfrak{F}) -hypercentral.*

Proof. Since $U \in \mathfrak{F}$, $U/\mathcal{C}_U(A/B) \in f(p)$. Therefore $U/\mathcal{C}_U(V) \in f(p)$. \square

Green and Hill have proved (see Doerk and Hawkes [7, Theorem B 6.17, p.136]) that if A/B is a p -chief factor of a p -soluble group U , then $A/B \in B_1(\mathbb{F}_p U)$. The following lemma generalises this.

Lemma 3.2. *Let A/B be a p -chief factor of the p -soluble group U . Then every composition factor of $F \otimes_{\mathbb{F}_p} (A/B)$ is in $B_1(FU)$.*

Proof. The largest p -nilpotent normal subgroup $O_{p'p}(U)$ is the intersection of the centralisers of the p -chief factors of U , so $O_{p'p}(U) \subseteq \mathcal{C}_U(A/B)$. If V is a composition factor of $F \otimes_{\mathbb{F}_p} (A/B)$, then $\mathcal{C}_U(V) \supseteq \mathcal{C}_U(A/B)$. By a theorem of Fong and Gaschütz, (see Doerk and Hawkes [7, Theorem B 4.22, p. 118]) an irreducible FU -module V is in the principal block if and only if $O_{p'p}(U) \subseteq \mathcal{C}_U(V)$. Therefore $V \in B_1(FU)$. \square

Lemma 3.3. *Let V be an irreducible FG -module in the principal block. Then V is (U, \mathfrak{F}) -hypercentral.*

Proof. Since $V \in B_1(G)$, there exists a chain of irreducible G -modules $V_0, \dots, V_n = V$ where V_0 is the trivial module, and modules X_1, \dots, X_n where X_i is a non-split extension of one of V_{i-1}, V_i by the other. If V is not \mathfrak{F} -central, then for some k , we have V_{k-1} \mathfrak{F} -central and V_k \mathfrak{F} -eccentric. But then, by Theorem 2.4, X_i cannot be indecomposable. \square

If \mathfrak{F} is the smallest saturated formation containing the soluble group G and $N = O_{p'p}$ is the largest p -nilpotent normal subgroup of G , then for $f(p)$, we may take the smallest formation containing G/N .

Lemma 3.4. *Let K be the $f(p)$ -residual of G . Then $K = N$.*

Proof. Let G be a minimal counterexample. Let A be a minimal normal subgroup of G . Suppose $A \subseteq O_{p'}$. Then $N/A = O_{p'p}(G/A)$ and the result holds for G/A and so also for G . It follows that $O_{p'}(G) = \{1\}$ and $A \subseteq N$. If $A \neq N$, again we have that the result holds for G/A and for G . Therefore N is the only minimal normal subgroup of G .

The class \mathfrak{N}^r of groups of nilpotent length r is a formation. If G has nilpotent length r , then G/N has nilpotent length $r - 1$ and it follows that $f(p) \subseteq \mathfrak{N}^{r-1}$. Thus the $f(p)$ -residual of G cannot be $\{1\}$ and so must be N , contrary to the assumption that G is a counterexample. \square

Theorem 3.5. *Let G be a finite soluble group and let F be a field of characteristic p . Let \mathfrak{F} be the smallest saturated formation containing G and let V be an irreducible FG -module. Then V is \mathfrak{F} -central if and only if $V \in B_1(FG)$.*

Proof. By Lemma 3.3, if $V \in B_1(FG)$, then V is \mathfrak{F} -central. Conversely, if V is \mathfrak{F} -central, then $\mathcal{C}_G(V) \supseteq O_{p'p}(G)$ and $V \in B_1(FG)$ by the Fong-Gaschütz Theorem [7, B 4.22, p.118]. \square

Theorem 3.5 does not need the full force of the assumption that \mathfrak{F} is the smallest saturated formation containing G , merely that $f(p)$ is minimal. Restrictions on the $f(q)$ for $q \neq p$ are irrelevant.

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